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On a safe capital stock for consumption maintenance in a convex model of stochastic growth

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In a stochastic one-sector optimal growth model with convex technology we show that there may be a critical "safe" capital stock for maintaining positive consumption over time. Consumption and capital paths are bounded away from zero with probability one if the initial stock is above this safe level, but are arbitrarily close to zero with positive probability if it is below. We derive verifiable sufficient conditions on technology and preferences under which the optimal policy generates such a critical stock. High risk aversion near zero, decreasing risk aversion and large spread of the technology shock are important factors behind this phenomenon.

Key words stochastic growth, concave production function, technology shock, critical stock, positive consumption, safe standard of conservation

JEL classification D60, D70, D90

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1 Introduction

An important purpose of economic growth theory is to understand the circumstances under which a positive level of consumption is maintained over time. Even when the growth path of a competitive market economy maximizes the intertemporal welfare of its agents, one may be justifiably concerned about dynamic social stability if the consumption possibilities are reduced to zero, or levels arbitrarily close to zero, over time. It is well known that the dynamic economic incentives of agents may lead to the latter outcome even when it is technologically feasible to sustain positive capital and consumption over time. When the production technology is convex i.e., marginal productivity of investment decreases with accumulation of capital, the return on investment is at its highest when the economy's capital stock is near zero. For such technology, one would expect that the incentive to deplete capital would be minimized near zero so that if a minimum positive level of capital and consumption over time is maintained from *some* initial capital stock, it would also be true from initial capital stocks that are close to zero. In other words, in a convex framework, one would expect that either capital and consumption may be depleted

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This paper is dedicated to the memory of Lionel McKenzie. We have benefited from the comments made by an anonymous referee of this journal.

to levels arbitrarily close to zero from all positive initial stocks, or the economy maintains a positive level of capital and consumption from all such stocks. This intuition is confirmed by results obtained in the deterministic one sector model of optimal economic growth where (see last part of Section 3) consumption and capital either decline to zero from all initial capital stocks, or converge to a positive steady state from all (positive) initial stocks. In this paper, we argue that in the *stochastic* version of the model, the incentive to expand capital and consumption may not be maximized near zero even though the technology is convex and marginal productivity of investment is maximized at zero. We show that there may be a critical "safe" stock such that for initial stocks below the safe level, levels of consumption and capital may get arbitrarily close to zero (with positive probability) while for initial stocks above the safe level, a minimum positive level of consumption and capital are maintained over time.

In the literature, the existence of such a critical stock has been typically associated with non-convexity in the technology (such as S-shaped production functions). With such non-convexity, productivity can be low when stocks are close to zero so that it may be optimal to deplete capital over time when the initial capital is small, but it may be optimal to sustain capital (and consumption) when the initial capital is large (and productivity of investment is higher).[1] While such non-convexities capture important features such as increasing returns to scale in production technology and "depensation" in biological production (in models of optimal resource management), much of the mainstream macroeconomic growth literature has focused on models where the aggregate technology is convex and marginal productivity of investment diminishes with capital accumulation. This paper indicates that with uncertainty, the optimal policy in such a convex model may display some of the qualitative features found in deterministic models when the technology is non-convex.

Our framework is the standard one sector optimal stochastic growth model with discounting where technology shocks are independent and identically distributed over time and have bounded support (Brock and Mirman 1972); in particular, we confine attention to the case where the technology shock is multiplicative and the marginal productivity at zero is finite. We derive sufficient conditions under which the optimal policy is characterized by a critical "safe" stock for capital and consumption to be bounded away from zero.

The important economic factors driving our result are the extent of risk and the degree of risk aversion of the representative agent. In the presence of risk, risk aversion creates an incentive to favor the certainty of current consumption against the uncertain stream of future consumption added by investment. When the spread of technological uncertainty is large, high risk aversion near zero depresses the incentive to invest so that capital and output decline under bad realizations of the technological shock; indeed, this may occur even if the productivity or return on investment is very high near zero. On the other hand, if risk aversion declines sharply with consumption, then for levels of initial stock that are somewhat higher (above a "safe" level), it may be optimal to invest enough so that capital and output expand even under the worst realization of the technology

1 See, among many others, Majumdar and Mitra (1982), Dechert and Nishimura (1983), Clark (1990), Kamihigashi and Roy (2007).

shock, and a minimum level of capital and consumption is maintained over time with probability one.

The literature on stochastic growth with convex technology characterizes conditions under which capital and consumption paths from *all* initial stocks are bounded away from zero (see, among others, Brock and Mirman 1972; Chatterjee and Shukayev 2008; Mitra and Roy 2010).[2] A smaller literature has focused on the possibility that capital and consumption paths are arbitrarily close to zero no matter how large the initial stock, and even if the marginal productivity at zero is infinitely large (see, Mirman and Zilcha 1976; Mitra and Roy 2010).[3] The literature on *non-convex* stochastic models of optimal resource management has discussed the existence of a critical safe stock for maintaining capital and consumption (referred to as a "safe standard of conservation"); see, among others, Olson and Roy (2000) and Mitra and Roy (2006). To the best of our knowledge, this is the first paper to demonstrate the possibility of such a critical stock in a convex model.

The paper is organized as follows. Section 2 outlines the model and some basic results. Section 3 discusses in detail the concept of maintaining consumption and the critical safe stock. Section 4 contains the main results of the paper. Proofs of the results are contained in Section 5.

2 The model

We consider an infinite-horizon one-good representative agent economy. Time is discrete and is indexed by $t = 0, 1, 2, \ldots$. The initial stock of output $y_0 > 0$ is given. At each date $t \geq 0$, the representative agent observes the current output stock $y_t \in \mathbb{R}_+$ and chooses the level of consumption c_t and current investment x_t, such that

$$c_t \geq 0,\ x_t \geq 0,\ c_t + x_t \leq y_t$$

This generates y_{t+1}, the output stock next period through the relation

$$y_{t+1} = r_{t+1} h(x_t)$$

where $h(.)$ is the deterministic component of the production function and r_{t+1} is a multiplicative random production shock realized at the beginning of period $(t+1)$. Given current output stock $y \geq 0$, the feasible set for consumption and input is denoted by $\Gamma(y)$; that is,

$$\Gamma(y) = \{(c, x) : c \geq 0, x \geq 0, c + x \leq y\}.$$

The following assumption is made on the sequence of random shocks:

[2] More general models of stochastic growth that allow for non-convexity have also addressed this question. See, among others, Olson and Roy (2000), Mitra and Roy (2006), Nishimura, Rudnicki, and Stachurski (2006).

[3] It is important to distinguish this literature from models of stochastic growth where technology shocks are "unbounded" so that the output generated by any level of investment, however large, may be arbitrarily small. Kamihigashi (2007) shows that if the marginal product at zero is finite, then *every feasible path* converges *almost surely* to zero, provided the random shocks are "sufficiently volatile".

(A.1) $\{r_t\}_{t=1}^{\infty}$ *is an independent and identically distributed random process defined on a probability space* (Ω, F, P), *where the marginal distribution function is denoted by* F. *The distribution* F *is concentrated*[4] *on a closed interval* $A = [\alpha, \beta] \subset \mathbb{R}_{++}$ *where* $0 < \alpha < \beta < \infty$, *and for all* $s \in (0, \beta)$,

$$F(\alpha + s) > 0, \; F(\beta - s) < 1.$$

Further, $E(r^{-p}) = \int_A r^{-p} dF(r)$ *is continuous in* p *on* $\mathbb{R}_{++}$.

The last part of (A.1) is a technical assumption that simplifies the conditions outlined in our main results.

The deterministic part of the production function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to satisfy the following:

(T.1) $h(x)$ *is concave in* x *on* $\mathbb{R}_+$.
(T.2) $h(0) = 0$.
(T.3) $h(x)$ *is strictly increasing and continuously differentiable on* $\mathbb{R}_+$, $h'(x) > 0$ for all $x \geq 0$.
(T.4) h *satisfies the following:*

$$h'(0) > \frac{1}{\alpha}. \tag{1}$$

Further:

$$\lim_{x \to \infty} \frac{\beta h(x)}{x} < 1.$$

Assumptions (T.1)–(T.3) are standard monotonicity, concavity and smoothness restrictions on production. In particular, (T.3) implies that marginal productivity is bounded on $\mathbb{R}_+$ and so $h'(0) < \infty$. The first part of assumption (T.4) ensures that it is feasible for capital and output to grow in a neighborhood of zero even under the most adverse realization of the random shock. The second part of the assumption implies that the technology exhibits bounded growth. There is a unique positive solution to the equation $\beta h(x) = x$; we denote this by K. Then,

$$\beta h(x) > x \text{ for all } x \in (0, K); \beta h(x) < x \text{ for all } x > K. \tag{2}$$

Let $\delta \in (0, 1)$ denote the utility discount factor. Given the initial stock $y_0 > 0$, the representative agent's objective is to maximize the expected value of the discounted sum of utilities from consumption:

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t)\right]$$

where u is the one period utility function from consumption.

[4] For this terminology, and its relation to the support of the distribution, see Marshall and Olkin (2010).

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. The utility function $u : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ satisfies the following restrictions:

(U.1) *u is strictly increasing, continuous and strictly concave on* $\mathbb{R}_+$ (on $\mathbb{R}_{++}$ if $u(0) = -\infty$); $u(c) \rightarrow u(0)$ as $c \rightarrow 0$.
(U.2) *u is twice continuously differentiable on* $\mathbb{R}_{++}$ with $u'(c) > 0$, $u''(c) < 0$ for all $c > 0$.
(U.3) $\lim_{c \rightarrow 0} u'(c) = +\infty$.

Assumptions (U.1) and (U.2) are standard. Note that we allow the utility of zero consumption to be $-\infty$. (U.3) requires that the utility function satisfy the Uzawa–Inada condition at zero.

Let $R(c)$ be the Arrow–Pratt measure of relative risk aversion at c defined by:

$$R(c) = -\frac{u''(c)c}{u'(c)}$$

Further, let $\underline{R}$, $\overline{R}$ be defined by:

$$\underline{R} = \liminf_{c \downarrow 0} R(c), \ \ \overline{R} = \limsup_{c \downarrow 0} R(c).$$

The *partial history* at date t is given by $h_t = (y_0, x_0, c_0, \ldots, y_{t-1}, x_{t-1}, c_{t-1}, y_t)$. A *policy* π is a sequence $\{\pi_0, \pi_1, \ldots\}$ where π_t is a conditional probability measure such that $\pi_t(\Gamma(y_t)|h_t) = 1$. A policy is *Markovian* if for each t, π_t depends only on y_t. A Markovian policy is *stationary* if π_t is independent of t. Associated with a policy π and an initial state y is an expected discounted sum of social welfare:

$$V_\pi(y) = E \sum_{t=0}^{\infty} \delta^t u(c_t),$$

where $\{c_t\}$ is generated by π, f in the obvious manner and the expectation is taken with respect to P. The *value function* $V(y)$ is defined on $\mathbb{R}_{++}$ by:

$$V(y) = sup\{V_\pi(y) : \pi \ is \ a \ policy\}.$$

Under assumption (T.4), it is easy to check that:

$$-\infty < V(y) < +\infty \quad \text{for all} \quad y > 0.$$

A policy, π^*, is *optimal* if $V_{\pi^*}(y) = V(y)$ for all $y > 0$. Standard dynamic programming arguments imply that there exists a unique optimal policy, that this policy is stationary and that the value function satisfies the functional equation:

$$V(y) = \sup_{x \in \Gamma(y)} [u(y-x) + \delta E\,(V(rh(x)))]. \tag{3}$$

It can be shown that $V(y)$ is continuous, strictly increasing and strictly concave on $\mathbb{R}_{++}$. Further, the maximization problem on the right hand side of (3) has a unique solution, denoted by $x(y)$. The stationary policy generated by the function $x(y)$ is the optimal policy and we refer to $x(y)$ as the *optimal input function* and $c(y) = y - x(y)$ as the *optimal* *consumption function.* Using standard arguments[5] in the literature, (U.3) can be used to show that:

Lemma 1

(i) For all $y > 0$, $x(y) > 0$ *and* $c(y) > 0$.
(ii) $x(y)$ *and* $c(y)$ *are continuous and strictly increasing in* y *on* $\mathbb{R}_+$.

Next, we note that the stochastic Ramsey–Euler equation holds:

Lemma 2 *For all* $y > 0$,

$$u'(c(y)) = \delta E[u'(c(rh(x(y))))rh'(x(y))]. \tag{4}$$

We confine attention to levels of initial stock y_0 lying in the set $(0, K]$.This implies that feasible consumption, investment and output in every period lie in the set $[0, K]$ with probability one. Since u is bounded above on the set $[0, K]$, we may without loss of generality assume that:

(U.4) $u(c) \leq 0$ for all $c > 0$.

Let $\underline{\eta}$ be the limiting first elasticity of the utility function at zero defined by

$$\liminf_{c \downarrow 0} \left[-\frac{u'(c)c}{u(c)} \right] = \underline{\eta}.$$

We assume that:

(U.5) $\underline{\eta} > 0$.

Assumptions (A.1), (T.1)–(T.4) and (U.1)–(U.5) are retained throughout the paper.

3 On the concept of safe stock for consumption maintenance

The central focus of this paper is the possibility of maintaining consumption over time; we are interested in knowing whether the economy's consumption path is uniformly bounded below by a strictly positive level of consumption with probability one. In view of Lemma 1, it is easy to see that this requires that investment and output are also uniformly bounded below almost surely by a strictly positive number. In particular, given $y_0 > 0$, suppose

$$\exists\, y' \in (0, y_0] \quad \text{such that} \quad \alpha h(x(y')) \geq y'. \tag{5}$$

Then, using Lemma 1, it is easy to check that the stochastic process of optimal output stocks $\{y_t(y_0, \omega)\}$ from initial stock y_0 defined by :

$$y_t(y_0, \omega) = \omega_t h(x(y_{t-1}(y_0, \omega))) \quad \textit{for } t \geq 1,$$

[5] See especially Brock and Mirman (1972) and Mirman and Zilcha (1975).

satisfies:

$$y_t(y_0, \omega) \geq y' \text{ almost surely for all } t \geq 0,$$

and therefore, optimal consumption $c_t(y_0, \omega) = c(y_t(y_0, \omega))$ satisfies:

$$c_t(y_0, \omega) \geq c' \text{ almost surely for all } t \geq 0,$$

where $c' = c(y') > 0$. On the other hand, if there is no $y' \in (0, y_0]$ for which (5) holds i.e.,

$$\alpha h(x(y)) < y \quad \text{for all} \quad y \in (0, y_0], \tag{6}$$

then output and consumption decline and approach zero if the worst shock occurs every period; in particular, for any $\epsilon > 0$ there exists T such that for all $t \geq T$,

$$\Pr\{\omega \in \Omega : c_t(y_0, \omega) < \epsilon\} > 0.$$

In other words, (5) is a necessary and sufficient condition to ensure that the lower bound of the support of optimal consumption over time is bounded away from zero.

Given the optimal policy function $x(y)$, there are (only) three possibilties:

(i) there is a sequence $\{y^n\}_{n=1}^{\infty}$, $y^n > 0$ for all n, such that $\{y^n\} \to 0$ and $\alpha h(x(y^n)) \geq y^n$ for all n;
(ii) $\alpha h(x(y)) < y$ for all $y > 0$;
(iii) there exists $y^* > 0$ such that $\alpha h(x(y)) < y$ for all $y \in (0, y^*)$ and $\alpha h(x(y^*)) \geq y^*$.

In configuration (i), (5) is satisfied for all $y_0 > 0$ so that independent of initial stock, the support of consumption over time is almost surely bounded away from zero; the economy always maintains a minimum positive level of consumption (though the minimum level itself may depend on the initial stock). Output and capital are also bounded away from zero with probability one. Note that configuration (i) includes the situation where capital and consumption grow near zero even under the worst realization of the random shock. The existing literature on stochastic growth contains various sufficient conditions on preferences and technology under which the optimal policy satisfies configuration (i); see, among others, Brock and Mirman (1972), Chatterjee and Shukayev (2008), Mitra and Roy (2006, 2010) and Olson and Roy (2000).

In configuration (ii), from any positive initial stock, capital and consumption necessarily decline under the worst realization of the random shock. In particular, (5) does not hold for any $y_0 > 0$ and the lower bound of the support of consumption converges to zero over time. It has been shown that in this case consumption and capital fall below any strictly positive level infinitely often *with probability one,* no matter how large the initial stock (Mitra and Roy 2007). While it is easy to see that configuration (ii) can arise if the marginal productivity at zero is sufficiently small relative to the discount rate, Mirman and Zilcha (1975) presented an example to show that the optimal policy may lead to configuration (ii) even if the marginal productivity is infinite at zero for all realizations of the random shock; Mitra and Roy (2010) develop necessary and sufficient conditions for this phenomenon and relate it to the nature of risk aversion displayed by the utility function.

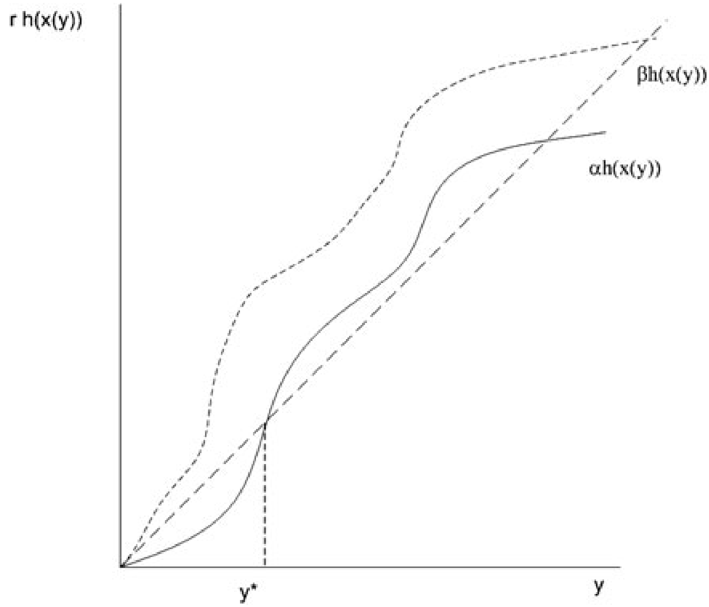

Figure 1 Configuration (iii).

The possibility of configuration (iii) in convex models of stochastic growth remains unexplored. In configuration (iii), capital and consumption decline under the worst shock if the initial stock is below a critical stock $y^* > 0$, but are bounded away from zero if the initial stock is above y^*. If $y_0 < y^*$then (6) holds so that capital and consumption may be arbitrarily close to zero (the lower bound of the support of consumption declines to zero over time). On the other hand, if $y_0 \geq y^*$ then (5) is satisfied and the consumption path is uniformly bounded below by $c(y^*)$ with probability one. The stock y^* is the minimum safe stock for maintaining positive consumption.

Figure1 illustrates configuration (iii). Note that in the figure, optimal output expands near zero under the best shock β; however, configuration (iii) as defined above imposes no restriction on the behavior of the economy under better realizations of the random shock.

In the deterministic version of the model, optimal policy generates either configuration (i) or (ii); configuration (iii) never arises.[6] For instance suppose that the distribution of the random shock is degenerate and, in particular, $r_t = 1$ with probability one. Then, capital and consumption grow near zero (converging to a positive steady state) and configuration (i) obtains if $h'(0) > \frac{1}{\delta}$. If $h'(0) < \frac{1}{\delta}$, the capital and consumption decline everywhere and converge to zero i.e., configuration (ii) obtains. If $h'(0) = \frac{1}{\delta}$, then configuration (ii) obtains unless h is linear in a neighborhood of zero (in which case, configuration (i) obtains.) We will show that configuration (iii) may however arise in the stochastic growth model, and this illustrates an important qualitative difference between stochastic and deterministic models.

[6] This may not hold if the utility from consumption depends on wealth or capital. See, Roy (2010).

4 Main results

In this section, we outline the main results of the paper. In particular, we develop conditions under which the optimal policy generates configuration (iii) described in the previous section i.e., there exists a critical level of initial stock below which capital and consumption may be arbitrarily close to zero over time and above which, a minimum positive level of capital and consumption is maintained over time almost surely.

4.1 Decline near zero under worst shock

In this subsection, we outline a sufficient condition under which output (and therefore, investment and consumption) declines under the worst realization of the random shock i.e., $\alpha h(x(y)) < y$ if the current stock y is sufficiently close to zero. This rules out the possibility of maintaining a positive level of capital and consumption with probability one from small stocks. Recall the definitions of $\underline{R}$, $\overline{R}$, the limits of relative risk aversion at zero, and $\underline{\eta}$, the lower limit of the first elasticity of the utility function at zero, in Section 2.

Proposition 1 *Suppose that* $1 < \underline{R} \leq \overline{R} < \infty$ *and*

$$\left[E\left\{\left(\frac{\alpha}{r}\right)^{\underline{R}-1}\right\}\delta\alpha h'(0)\right]^{\frac{1}{\overline{R}}} < \left[1 - \frac{1}{\alpha h'(0)}\right][1-\delta]^{\frac{1}{\underline{\eta}}}. \tag{7}$$

Then, there exists $\widetilde{y} > 0$ *such that* $h(x(y)) < y$ *for all* $y \in (0, \widetilde{y})$.

Condition (7) in Proposition 1 provides a verifiable sufficient condition on the stochastic technology and preferences under which decline under worst shock occurs near zero. Note that for $\underline{R} \geq 2$, $E\{(\frac{\alpha}{r})^{\underline{R}-1}\}$ is likely to be arbitrarily small if realizations r sufficiently larger than α occur with significant probability and in that case, condition (7) is likely to be satisfied even if $\alpha h'(0) > \frac{1}{\delta}$ i.e., the technology is "delta-productive" even under the worst realization of the random shock. To understand more clearly the kind of restrictions implied by this sufficient condition and when it can be satisfied, we consider a parametric family of utility functions and a specific distribution of the random shock.

Example 1 Consider the following utility function u that exhibits decreasing relative risk aversion

$$u(c) = \lambda\frac{c^{1-\theta}}{1-\theta} + (1-\lambda)\frac{c^{1-\sigma}}{1-\sigma} - M, \tag{8}$$

where

$$\theta > \sigma > 0, \theta > 2, \sigma \neq 1, 0 < \lambda < 1,$$

and

$$M > \lambda\frac{K^{1-\theta}}{1-\theta} + (1-\lambda)\frac{K^{1-\sigma}}{1-\sigma}.$$

Then, $u(c) < 0$ on $(0, K]$ and $u(c) \to -\infty$ as $c \to 0$. Further,

$$\underline{\eta} = \liminf_{c \downarrow 0} \left[-\frac{u'(c)c}{u(c)} \right] = \theta - 1,$$

and

$$R(c) = -\frac{u''(c)c}{u'(c)} = \frac{[\lambda\theta + \sigma(1-\lambda)c^{\theta-\sigma}]}{\lambda + (1-\lambda)c^{\theta-\sigma}}$$

so that $R(c)$ is strictly decreasing in c on $\mathbb{R}_{++}$,

$$\overline{R} = \underline{R} = \lim_{c \to 0} R(c) = \theta.$$

Let r be uniformly distributed on the interval $[\alpha, \beta]$. Then,

$$E\left\{ \left(\frac{\alpha}{r}\right)^{\underline{R}-1} \right\} = \frac{\alpha^{\theta-1}}{\beta-\alpha} \int_{\alpha}^{\beta} r^{1-\theta} dr = \frac{\alpha}{(\theta-2)(\beta-\alpha)} \left[1 - \left(\frac{\alpha}{\beta}\right)^{\theta-2} \right]$$

and condition (7) reduces to the requirement that

$$\left[\frac{\alpha}{(\theta-2)(\beta-\alpha)} \left(\frac{1 - \left(\frac{\alpha}{\beta}\right)^{\theta-2}}{(1-\delta)^{\frac{\theta}{\theta-1}}} \right) \right]^{\frac{1}{\theta}} < \frac{1}{\delta\alpha h'(0)} \left[1 - \frac{1}{\alpha h'(0)} \right]. \tag{9}$$

Given other parameters, condition (9) is satisfied if α is small enough relative to β (return on investment is sufficiently "risky") and, as assumed above, $\theta > 2$ (the risk aversion at zero is large enough). For instance, suppose $\theta = 3$ and $h'(0) = \frac{2}{\alpha}$; then, condition (9) is satisfied if

$$\frac{\alpha}{\beta} < \left[\frac{\sqrt{1-\delta}}{4\delta} \right]^3.$$

If $\delta = 0.9$, this is satisfied for $\alpha = 0.01$, $\beta = 17.3$.

It is worth noting that though the utility function used in the above example is one that exhibits DRRA (decreasing relative risk aversion), the DRRA property plays no role in Proposition 1 or in the phenomenon illustrated in the example; all that is required of the utility function is that the degree of risk aversion near zero should be large enough.

4.2 Maintaining capital and consumption from some stock

In this subsection, we provide a condition under which there is a safe stock $s > 0$ such that $\alpha h(x(s)) \geq s$; if the initial stock lies above this threshold then the consumption path is almost surely bounded below by $c(s) > 0$ and the economy sustains a minimum positive level of consumption, capital and output with probability one.

Using (1), (2) and the concavity of h, there exists $z_1 \in (0, K)$ such that

$$\alpha h(z_1) = z_1, \ \frac{h(x)}{x} > \frac{1}{\alpha} \quad \text{for all} \quad x \in (0, z_1).$$

For any $y \in (0, z_1)$, let $x_y \in (0, y)$ be defined by

$$\alpha h(x_y) = y \tag{10}$$

i.e., $x_y = h^{-1}(\frac{y}{\alpha})$. Further, for $y \in (0, z_1)$, let $\overline{R}(y)$ be defined by:

$$\overline{R}(y) = \max\{R(c) : y - x_y \leq c \leq \beta y\}$$

where, as before, $R(c)$ is the Arrow–Pratt measure of relative risk aversion.

Proposition 2 *Suppose there exists $s \in (0, z_1)$ such that*

$$\delta \alpha h'(x_s) \left(1 - \frac{x_s}{s}\right)^{\overline{R}(s)} E\left[\left(\frac{\alpha}{r}\right)^{\overline{R}(s)-1}\right] \geq 1. \tag{11}$$

Then $\alpha h(x(s)) \geq s$.

Condition (11) in Proposition 2 is a modified "delta-productivity" condition that requires the discounted marginal productivity to be larger than one by a factor that depends on the distribution of shocks and the degree of risk aversion. Among other things, the lower the degree of risk aversion the more likely that this condition is satisfied.

4.3 Critical stock for maintaining capital and consumption

If the requirements of Propositions 1 and 2 are satisfied simultaneously, then the optimal policy is one that exhibits decline of capital and consumption under worst shock when initial stock is near zero but allows maintenance of a positive level of consumption and capital if the initial stock exceeds a threshold level s. In that case, if

$$y^* = \inf\{s : \alpha h(x(s)) \geq s\},$$

then $y^* > 0$; y^* is a critical stock in the sense that $\alpha h(x(y)) < y$ for all $y < y^*$ and $\alpha h(x(y^*)) = y^*$; consumption over time may be arbitrarily close to zero with positive probability if $y_0 < y^*$, but if $y_0 \geq y^*$, the consumption path is almost surely uniformly bounded below by $c(y^*) > 0$. In other words, configuration (iii) described in Section 3 obtains.

The question that arises is whether and when both condition (7) in Proposition 1 and (11) in Proposition 2 can be satisfied simultaneously. We have noted earlier that given other parameters, condition (7) is satisfied if the relative risk aversion near zero is large and the spread of the shock is sufficiently large. On the other hand, given the distribution of the random shock, condition (11) is likely to be satisfied if relative risk aversion is sufficiently small and productivity is not too small at an intermediate level of consumption and investment. This suggests that both (7) and (11) may hold if the distribution of the random shock is sufficiently spread out, risk aversion is high at zero but declines sharply as consumption is increased while productivity declines relatively slowly as investment is increased.

In Example 1, we considered a specific family of decreasing relative risk aversion utility function and for the case of uniformly distributed shocks, we outlined a simple condition (9) on the parameters under which (7) and the conclusion of Proposition 1 holds. We now refine this example further by choosing a specific production function h and show that under certain parametric restrictions, both conditions (7) and (11) can be satisfied by the same economy so that the conclusions of both Propositions 1 and 2 hold.

Example 2 Consider the economy specified in Example 1. Further, suppose that

$$h(x) = \gamma x, \quad \text{for} \quad x \in (0, \overline{x})$$

where $\gamma > 2$ and $\frac{\gamma}{\gamma-1} < \overline{x}$. For $x \geq \overline{x}$, $h(x)$ can be any increasing concave extension that satisfies assumptions $(T.1) - (T.4)$. Then, condition (9) in Example 1 (which is sufficient to ensure that the conclusion of Proposition 1 holds) reduces to the requirement that

$$\left[\frac{\alpha}{(\theta-2)(\beta-\alpha)}\left(\frac{1-\left(\frac{\alpha}{\beta}\right)^{\theta-2}}{(1-\delta)^{\frac{\theta}{\theta-1}}}\right)\right]^{\frac{1}{\theta}} < \frac{1}{\delta\alpha\gamma}\left[1-\frac{1}{\alpha\gamma}\right]. \tag{12}$$

and, given other parameters, this is satisfied if $\beta - \alpha$ is large enough in which case the economy declines under the worst shock in a neighborhood of zero. Note that this imposes no restriction on the value of parameters λ and σ of the utility function. Assume that $\beta - \alpha$ is large enough for (12) to hold and that, further, the mean $\left(\frac{\alpha+\beta}{2}\right)$ of the distribution of shocks is large enough so that

$$\delta\left(\frac{\alpha+\beta}{2}\right)\gamma > 1. \tag{13}$$

We will now show that if $\lambda \in (0, 1)$, $\sigma \in (0, 1)$ are close enough to zero, then condition (11) is also satisfied for some $s > 0$. In particular, choose

$$s = \frac{\gamma}{\gamma - 1}.$$

Then, $x_s = h^{-1}(s) = \frac{1}{\gamma-1}$ and

$$s - x_s = 1.$$

Recall that the utility function (8) exhibits decreasing relative risk aversion and

$$R(c) = \frac{[\lambda\theta + \sigma(1-\lambda)c^{\theta-\sigma}]}{\lambda + (1-\lambda)c^{\theta-\sigma}}.$$

Thus,

$$\begin{aligned}\overline{R}(s) &= \max\{R(c) : s - x_s \leq c \leq \beta s\}\\ &= R(s - x_s) = R(1) = \lambda\theta + \sigma(1-\lambda).\end{aligned}$$

Now,

$$\begin{aligned}E\left[\left(\frac{\alpha}{r}\right)^{\overline{R}(s)-1}\right] &= E\left[\left(\frac{\alpha}{r}\right)^{R(1)-1}\right]\\ &= \frac{\alpha}{(R(1)-2)(\beta-\alpha)}\left[1-\left(\frac{\alpha}{\beta}\right)^{R(1)-2}\right]\end{aligned}$$

Condition (11) is satisfied at $s = 1$ if,

$$\delta\alpha\gamma\left(1-\frac{1}{\gamma}\right)^{R(1)}\frac{\alpha}{(R(1)-2)(\beta-\alpha)}\left[1-\left(\frac{\alpha}{\beta}\right)^{R(1)-2}\right] \geq 1. \tag{14}$$

Observe that as $\lambda \to 0$ and $\sigma \to 0$, $R(1) \to 0$ and the left hand side of the inequality in (14) converges to

$$\delta\left(\frac{\alpha+\beta}{2}\right)\gamma > 1$$

under assumption (13). Thus, given all other parameters (that satisfy the restrictions imposed earlier), there exists $\lambda \in (0, 1)$, $\sigma \in (0, 1)$ close enough to zero so that (14) holds i.e., condition (11) is satisfied at $s = 1$.

Though the above example is based on a DRRA (decreasing relative risk aversion) utility function, the fact that risk aversion is monotonically decreasing is not important for the phenomenon that is being illustrated. The existence of a critical safe stock requires high degree of risk aversion near zero and moderately low degree of risk aversion at intermediate levels of consumption; whether risk aversion is actually monotonic plays no role.

5 Proofs

Proof of Proposition 1 Using (7), there exists $\lambda \in (\frac{1}{\alpha h'(0)}, 1)$, such that

$$\left[E\left\{\left(\frac{\alpha}{r}\right)^{\underline{R}-1}\right\}\delta\alpha h'(0)\right]^{\frac{1}{\underline{R}}} < (1-\lambda)(1-\delta)^{\frac{1}{\underline{\eta}}}.$$

Further, there exists $\xi \in (\frac{1}{\underline{R}}, 1)$ close enough to 1 such that:

$$(1-\lambda)(1-\delta)^{\frac{1}{\underline{\eta}}} > \frac{1}{\xi}\left[E\left\{\left(\frac{\alpha}{r}\right)^{\xi\underline{R}-1}\right\}\delta\alpha h'(0)\right]^{\frac{\xi}{\underline{R}}}. \tag{15}$$

Step 1: A lower bound on the propensity to consume near zero. Let

$$q = \xi(1-\lambda)(1-\delta)^{\frac{1}{\underline{\eta}}} \tag{16}$$

We will show that there exists $\widehat{y} > 0$ such that

$$c(y) \geq qy, \qquad \text{for all} \;\; y \in (0, \widehat{y}). \tag{17}$$

First, observe that as $\lambda > \frac{1}{\alpha h'(0)}$ and $\frac{h(x)}{x} \uparrow h'(0)$ as $x \downarrow 0$, there exists $y^1 > 0$, such that

$$\frac{\alpha h(x)}{x} > \frac{1}{\lambda} \quad \textit{for all} \quad x \in (0, y^1). \tag{18}$$

For any $y \in (0, y^1)$, using (18),

$$\frac{h^{-1}\left(\frac{y}{\alpha}\right)}{y} < \lambda$$

and therefore, for all $y \in (0, y^1)$

$$y - h^{-1}\left(\frac{y}{\alpha}\right) = y\left[1 - \frac{h^{-1}\left(\frac{y}{\alpha}\right)}{y}\right] > y[1-\lambda]. \tag{19}$$

Since $(1-\delta)^{\frac{1}{\underline{\eta}}} > \xi(1-\delta)^{\frac{1}{\underline{\eta}}}$, we can choose $\nu \in (0,1)$, close enough to 1, such that

$$(1-\delta)^{\frac{1}{\nu\underline{\eta}}} > \xi(1-\delta)^{\frac{1}{\underline{\eta}}}. \tag{20}$$

Using assumption (U.5), there exists $y^2 > 0$, such that

$$\left[-\frac{u'(c)c}{u(c)}\right] \geq \nu\underline{\eta}, \qquad \text{for all} \qquad c \in (0, y^2).$$

i.e., for all $c \in (0, y^2)$,

$$\frac{u'(c)}{u(c)} = \frac{d}{dc}[\ln(-u(c))] \leq -\nu\underline{\eta}\frac{d}{dc}[\ln c]. \tag{21}$$

For any $A > 1$, $c' \in (0, \frac{y^2}{A})$, integrating both sides of (21) from c' to Ac', we have

$$\ln(-u(Ac')) - \ln(-u(c')) \leq -\nu\underline{\eta}(\ln Ac' - \ln c')$$

so that

$$\frac{u(Ac')}{u(c')} \leq A^{-\nu\underline{\eta}}. \tag{22}$$

Set

$$\widehat{y} = \min\left\{y^1, \frac{y^2}{1-\lambda}\right\}. \tag{23}$$

We now establish (17). Suppose to the contrary that

$$c(y_0) < qy_0, \qquad \text{for some} \qquad y_0 \in (0, \widehat{y}). \tag{24}$$

As the deterministic investment sequence where an amount $h^{-1}(\frac{y_0}{\alpha})$ is invested every period (and the rest is consumed) is feasible from initial stock y_0,

$$\begin{aligned} V(y_0) &\geq \frac{u\left(y_0 - h^{-1}\left(\frac{y_0}{\alpha}\right)\right)}{1-\delta} \\ &> \frac{u(y_0(1-\lambda))}{1-\delta}, \text{ using (19).} \end{aligned} \tag{25}$$

As $u(.) \leq 0$,

$$\begin{aligned} V(y_0) &\leq u(c(y_0)) \\ &< u(qy_0), \text{ using (24).} \end{aligned} \tag{26}$$

From (25) and (26),

$$\frac{u(y_0(1-\lambda))}{1-\delta} < u(qy_0)$$

so that,

$$\frac{u(y_0(1-\lambda))}{u(qy_0)} > (1-\delta) \tag{27}$$

(the inequality reverses as $u(qy_0) < 0$). Observe that (using (16)), $q < (1-\lambda)$, so that using (22) and (23)

$$\frac{u(y_0(1-\lambda))}{u(qy_0)} < \left(\frac{1-\lambda}{q}\right)^{-\nu\underline{\eta}} = \left[\frac{q}{1-\lambda}\right]^{\nu\underline{\eta}}$$

so that using (27) we have, $(1-\delta) < [\frac{q}{1-\lambda}]^{\nu\underline{\eta}}$ i.e.,

$$q > (1-\lambda)(1-\delta)^{\frac{1}{\nu\underline{\eta}}}. \tag{28}$$

However, from (16) and (20), we have:

$$q = \xi(1-\lambda)(1-\delta)^{\frac{1}{\underline{\eta}}} \leq (1-\lambda)(1-\delta)^{\frac{1}{\nu\underline{\eta}}},$$

which contradicts (28). This completes Step 1.

Step 2: Decline (under Worst Shock) Near Zero. We now establish the main claim of the proposition. Using the definition of $\underline{R}$, $\overline{R}$ in Section 2, there exists $\overline{y} \in (0, \widehat{y})$, such that:

$$\frac{1}{\xi}\overline{R} \geq R(c) \geq \xi\underline{R} \qquad \text{for all} \qquad c \in (0, \overline{y}). \tag{29}$$

Let

$$\widetilde{y} = \frac{\alpha}{\beta}\overline{y}. \tag{30}$$

We are ready to show that $\alpha h(x(y)) < y$ for all $y \in (0, \widetilde{y})$. Suppose not. Then, there exists $y \in (0, \widetilde{y})$ such that

$$\alpha h(x(y)) \geq y. \tag{31}$$

From the Ramsey-Euler Equation(4):

$$\begin{aligned} u'(c(y)) &= \delta E[u'(c(rh(x(y))))rh'(x(y))] \\ &\leq \delta E\left[u'\left(c\left(\frac{r}{\alpha}y\right)\right)r\right]h'(x(y)), \text{ using (31),} \end{aligned}$$

so that

$$\frac{1}{\delta h'(x(y))} \leq E\left[r\frac{u'\left(c\left(\frac{r}{\alpha}y\right)\right)}{u'(c(y))}\right]$$

Using Step 1, $c(\frac{r}{\alpha}y) \geq q\frac{r}{\alpha}y$ so that $u'(c(\frac{r}{\alpha}y)) \leq u'(q\frac{r}{\alpha}y)$; further, $c(y) \leq y$, so that $u'(c(y)) \geq u'(y)$, and therefore

$$\begin{aligned} \frac{1}{\delta h'(x(y))} &\leq E\left[r\frac{u'\left(q\frac{r}{\alpha}y\right)}{u'(y)}\right] = E\left[r\left\{\frac{u'\left(q\frac{r}{\alpha}y\right)}{u'\left(\frac{r}{\alpha}y\right)}\right\}\left\{\frac{u'\left(\frac{r}{\alpha}y\right)}{u'(y)}\right\}\right] \\ &\leq E\left[r\left\{\frac{1}{q}\right\}^{\frac{\overline{R}}{\xi}}\left\{\frac{\alpha}{r}\right\}^{\xi\underline{R}}\right]. \end{aligned} \tag{32}$$

The last inequality follows from (29), the fact that $\frac{\beta}{\alpha}y < \overline{y}$ (using (30)), and Lemma 4 in Mitra and Roy (2010). From (32)

$$q^{\frac{\overline{R}}{\xi}} \leq E\left[\left(\frac{\alpha}{r}\right)^{\xi\underline{R}-1}\right]\delta\alpha h'(0)$$

i.e.,

$$q \leq \left[E\left\{\left(\frac{\alpha}{r}\right)^{\xi\underline{R}-1}\right\}\delta\alpha h'(0)\right]^{\frac{\xi}{\overline{R}}} \tag{33}$$

Using the expression for q in (16) we have

$$\xi(1-\lambda)(1-\delta)^{\frac{1}{\eta}} \leq \left[E\left\{\left(\frac{\alpha}{r}\right)^{\xi\underline{R}-1}\right\}\delta\alpha h'(0)\right]^{\frac{\xi}{\overline{R}}}$$

which contradicts (15). The proof is complete. □

Proof of Proposition 2 Suppose to the contrary that

$$\alpha h(x(s)) < s. \tag{34}$$

Then,

$$x(s) < x_s, \; c(s) > s - x_s. \tag{35}$$

This implies that

$$h'(x(s)) \geq h'(x_s), \; u'(c(s)) < u'(s - x_s) \tag{36}$$

From the Ramsey-Euler Equation(4):

$$u'(c(s)) = \delta E[u'(c(rh(x(s))))rh'(x(s))]$$

Using (36),

$$\begin{aligned} u'(s - x_s) &> \delta E[u'(c(rh(x(s))))rh'(x_s)] \\ &\geq \delta E\left[u'\left(c\left(\frac{r}{\alpha}s\right)\right) rh'(x_s)\right], \text{ using (34)} \\ &\geq \delta E\left[u'\left(\frac{r}{\alpha}s\right) rh'(x_s)\right] \end{aligned}$$

i.e.,

$$\delta E\left[\frac{u'\left(\frac{r}{\alpha}s\right)}{u'(s - x_s)} rh'(x_s)\right] < 1 \tag{37}$$

Using Lemma 4 in Mitra and Roy (2010), we have,

$$\frac{u'\left(\frac{r}{\alpha}s\right)}{u'(s - x_s)} = \frac{u'\left(\frac{r}{\alpha}s\right)}{u'\left(\left(1 - \frac{x_s}{s}\right)s\right)} \geq \left(\frac{1 - \frac{x_s}{s}}{\frac{r}{\alpha}}\right)^{\overline{R}(s)}$$

and using this in (37) yields:

$$\delta\alpha h'(x_s)\left(1 - \frac{x_s}{s}\right)^{\overline{R}(s)} E\left[\left(\frac{\alpha}{r}\right)^{\overline{R}(s)-1}\right] < 1. \tag{38}$$

which contradicts (11). □

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